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COMMENT

Hamiltonian and recursion operator for the reduced Maxwell–Bloch equations

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Abstract. The Hamiltonian of the reduced Maxwell–Bloch equations (RMBE), an integrable nonlinear evolution equation, is derived. The recursion operator, its inverse and also the hierarchy of higher-order RMBE are obtained.

The reduced Maxwell–Bloch equations (RMBE) have been derived and discussed in detail by Eilbeck *et al* (1973) and Bullough *et al* (1979). They represent the propagation of a short laser pulse in a rarefied medium of two level atoms. The equations are

$$\sigma_t(x,t) = -\int_{-\infty}^{x} v(x_1,t)\,dx_1, \tag{1}$$

$$u_x(x,t) = -\mu v(x,t), \qquad v_x(x,t) = \sigma_x(x,t)w(x,t) + \mu u(x,t),$$
$$w_x(x,t) = -\sigma_x(x,t)v(x,t). \tag{2}$$

$\sigma_x(x,t) = E(x,t)$ is the electric field. The boundary conditions are

$$w(x,t) \to -1 \text{ as } |x| \to \infty, \qquad u(x,t),\ v(x,t) \to 0 \text{ as } |x| \to \infty. \tag{3}$$

However, the above authors have not found the Hamiltonian for the RMBE. Here we obtain the Hamiltonian and the recursion operator for the RMBE and also generate the higher-order RMBE.

From (2) we get the differential equations

$$\begin{aligned} v_x + iw_x &= -i\sigma_x(v+iw) - \mu^2 \int_{-\infty}^{x} v\,dx_1, \\ v_x - iw_x &= i\sigma_x(v-iw) - \mu^2 \int_{-\infty}^{x} v\,dx_1. \end{aligned} \tag{4}$$

We solve these equations using the boundary conditions (3) and we obtain

$$\begin{aligned} v(x,t) = -\sin\sigma(x,t) - \mu^2\Big(&\cos\sigma(x,t)\int_{-\infty}^{x} dx_1 \cos\sigma(x_1,t)\int_{-\infty}^{x_1} v(x_2,t)\,dx_2 \\ &+ \sin\sigma(x,t)\int_{-\infty}^{x} dx_1 \sin\sigma(x_1,t)\int_{-\infty}^{x_1} v(x_2,t)\,dx_2\Big). \end{aligned} \tag{5}$$

Iterating (5) and substituting in (1) we get

$$\sigma_t(x,t)=\sum_{n=0}^{\infty}(-1)^n\mu^{2n}[T_s^{-n}(\sigma(x,t))]\left(\int_{-\infty}^{x}\sin\sigma(x_{2n+1},t)\,\mathrm{d}x_{2n+1}\right), \tag{6}$$

where

$$\begin{aligned}T_s^{-1}(\sigma(x,t))=&\int_{-\infty}^{x}\mathrm{d}x_1\sin\sigma(x_1,t)\int_{-\infty}^{x_1}\mathrm{d}x_2\sin\sigma(x_2,t)\\&+\int_{-\infty}^{x}\mathrm{d}x_1\cos\sigma(x_1,t)\int_{-\infty}^{x_1}\mathrm{d}x_2\cos\sigma(x_2,t)\end{aligned} \tag{7}$$

is the operator which acting on $\int_{-\infty}^{x_2}\sin\sigma(x_3,t)\,\mathrm{d}x_3$ successively generates the higher sine-Gordon (SG) equations (Aiyer 1983). Thus the coefficient of μ^{2n} on the RHS of (6) is the nth-order SG equation. The Hamiltonian $\bar{H}_{2n+1}$ of the nth-order SG equation has been derived by Sasaki and Bullough (1981): the first few are

$$\bar{H}_1=\int_{-\infty}^{\infty}(1-\cos\sigma)\,\mathrm{d}x,$$

$$\begin{aligned}\bar{H}_3=&\int_{-\infty}^{\infty}\sin\sigma\,\mathrm{d}x\int_{-\infty}^{x}\cos\sigma\,\mathrm{d}x_1\int_{-\infty}^{x_1}\sin\sigma\,\mathrm{d}x_2\\&+\int_{-\infty}^{\infty}\cos\sigma\,\mathrm{d}x\int_{-\infty}^{x}\sin\sigma\,\mathrm{d}x_1\int_{-\infty}^{x_1}\sin\sigma\,\mathrm{d}x_2.\end{aligned} \tag{8}$$

The Hamiltonian of the RMBE is therefore

$$H_{\mathrm{RMB}}=\sum_{n=0}^{\infty}(-1)^n\mu^{2n}\bar{H}_{2n+1}. \tag{9}$$

2. Recursion operator for RMBE

The AKNS scattering equation for the RMBE (1) is

$$\psi_{1x}+\mathrm{i}k\psi_1=q\psi_2,\qquad \psi_{2x}-\mathrm{i}k\psi_2=r\psi_1, \tag{10}$$

with $q(x,t)=-r(x,t)=\frac{1}{2}\sigma_x(x,t)$. This is also the scattering equation for the SG equation

$$\sigma_t(x,t)=\int_{-\infty}^{x}\sin\sigma(x_1,t)\,\mathrm{d}x_1. \tag{11}$$

Therefore the operator (Aiyer 1983)

$$T_s(\sigma)=\partial^2/\partial x^2+\sigma_x(x,t)\int_{-\infty}^{x}\mathrm{d}x_1\,\sigma_{x_1}(x_1,t)\,\partial/\partial x_1 \tag{12}$$

and its inverse (7), which are the recursion operators for the SG equation, are also the recursion operators for the RMBE (1). We have verified that (12) is a recursion operator for the RMBE. Now

$$T_s^{-1}(\sigma)\{\sigma_x\}=\int_{-\infty}^{x}\sin\sigma(x_1,t)\,\mathrm{d}x_1, \tag{13}$$

but we want a recursion operator $T_{\mathrm{RMB}}(\sigma)$ for the RMBE such that

$$T^{-1}_{\mathrm{RMB}}(\sigma)\{\sigma_x\} = -\int_{-\infty}^{x} v(x_1, t)\,\mathrm{d}x_1, \tag{14}$$

the RHS of equation (1), so that, in analogy with the SG equation, $T^{-n}_{\mathrm{RMB}}(\sigma)$ acting on σ_x will generate the higher-order RMBE.

From equations (2) and the boundary conditions (3) we have

$$v_x(x, t) + \sigma_x(x, t)\int_{-\infty}^{x} \sigma_{x_1}(x_1, t)v(x_1, t)\,\mathrm{d}x_1 = -\mu^2\int_{-\infty}^{x} v(x_1, t)\,\mathrm{d}x_1 - \sigma_x(x, t), \tag{15}$$

which can be written as

$$(T_s(\sigma) + \mu^2)\left\{\int_{-\infty}^{x} v(x_1, t)\,\mathrm{d}x_1\right\} = -\sigma_x(x, t). \tag{16}$$

Therefore if we define

$$T_{\mathrm{RMB}}(\sigma) = T_s(\sigma) + \mu^2 \tag{17}$$

which is a recursion operator for RMBE if $T_s(\sigma)$ is, we have

$$T^{-1}_{\mathrm{RMB}}(\sigma)\{\sigma_x(x, t)\} = -\int_{-\infty}^{x} v(x_1, t)\,\mathrm{d}x_1. \tag{18}$$

$T^{-1}_{\mathrm{RMB}}(\sigma)$ is easily evaluated as an infinite series using (17):

$$\begin{aligned} T^{-1}_{\mathrm{RMB}}(\sigma) &= [T_s(\sigma)(1 + T_s^{-1}(\sigma)\mu^2)]^{-1} \\ &= T_s^{-1}(\sigma) - \mu^2 T_s^{-2}(\sigma) + \mu^4 T_s^{-3}(\sigma) - \ldots. \end{aligned} \tag{19}$$

Using (18) and (19) and substituting in equation (1) we obtain (6).

The higher-order RMBE are

$$\sigma_t(x, t) = T^{-1}_{\mathrm{RMB}}(\sigma)\{\sigma_x(x, t)\} \tag{20}$$

and $T^{-n}_{\mathrm{RMB}}(\sigma)$ can be easily evaluated from (17) and the corresponding Hamiltonian written down.

Equations (6), (17) and (19) reduce to the results of the SG equation when $\mu = 0$ as they should.

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